ValsCalc

Chapter 7 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 7: Further Applications of Integration.

7.1: Arc Length
7.2: Surface Areas of Revolution
7.3: Consumer Surplus and Producer Surplus
7.4: Moments and Centers of Mass
7.5: Hydrostatics
7.6: Probability

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

Visit valscalc.com to access a free online calculus textbook and additional multiple-choice and free-response practice exams.

Further Applications of Integration

Number of Questions—16

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points Earned	Points Available
Short Questions		70
Question 15		15
Question 16		15
TOTAL		**100**

Short Questions

1. A supply curve is $y = 4\sqrt[3]{x+8}$, and the market price is \$12. Calculate the producer surplus. *(5 pts.)*

2. Calculate the surface area generated by rotating the arc given by $y = \sqrt{25-x^2}$, $-3 \leqslant x \leqslant 3$, about the x-axis. *(5 pts.)*

3. Calculate the arc length of $x = \frac{1}{3}(y+1)^{3/2}$ from $(0,-1)$ to $(\frac{8}{3},3)$. *(5 pts.)*

4. A randomly selected number X follows the following probability density function. *(5 pts.)*

$$f(x) = \begin{cases} 3x^2 e^{-x^3} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that X is greater than 1 ?

5. The portion of the graph of $y = \frac{1}{2}x^2$ from $(0,0)$ to $(2,1)$ is rotated about the y-axis. Calculate the surface area of revolution by integrating with respect to y. *(5 pts.)*

6. Find the mean of the following probability density function. *(5 pts.)*

$$g(x) = \begin{cases} \frac{1}{2}e^{x} & 0 \leqslant x \leqslant \ln 3 \\ 0 & \text{otherwise} \end{cases}$$

7. A rectangle of width 4 meters and height 2 meters is submerged in water such that the top side is 3 meters below the water's surface. Calculate the hydrostatic force acting on the rectangle. (The specific weight of water is 9800 newtons per cubic meter.) *(5 pts.)*

8. At a call center, 60% of calls are answered within 3 minutes. Assuming an exponential probability density function, calculate the exact probability that a customer is helped within 5 minutes. *(5 pts.)*

9. Set up an integral for the hydrostatic force acting on an equilateral triangle of side lengths 2 meters whose top vertex lies 1 meter beneath the water's surface.

10. What is the centroid of the region bounded by the graphs of $y = 4\cos x$, $y = 1$, $x = -\dfrac{\pi}{3}$, and $x = \dfrac{\pi}{3}$? *(5 pts.)*

11. Determine an arc length function for the graph of $y = 8 + 4t^{3/2}$ beginning at $(0, 8)$. (*5 pts.*)

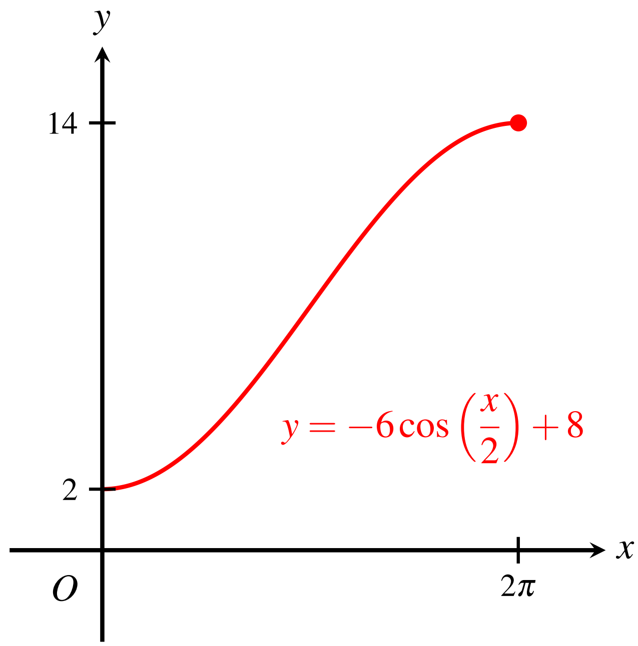
12. Consider the region bounded by the graphs of $x = 10 - y^2$ and $x = 1$. Find the y-coordinate of the centroid, and set up an expression containing integrals that equals the x-coordinate of the centroid. (*5 pts.*)

13. Write integral expressions that represent the centroid of the region bounded by $y = 7 - x^2$ and $y = x + 1$. *(5 pts.)*

14. A circular metal disk of radius 2 feet is submerged such that the disk's center sits 6 feet beneath the water's surface. Set up, but do not evaluate, an integral that equals the hydrostatic force acting on the disk. *(5 pts.)*

Long Questions

15. The graph of $y = -6\cos\left(\frac{x}{2}\right) + 8$ for $0 \leqslant x \leqslant 2\pi$ is shown in the following figure.



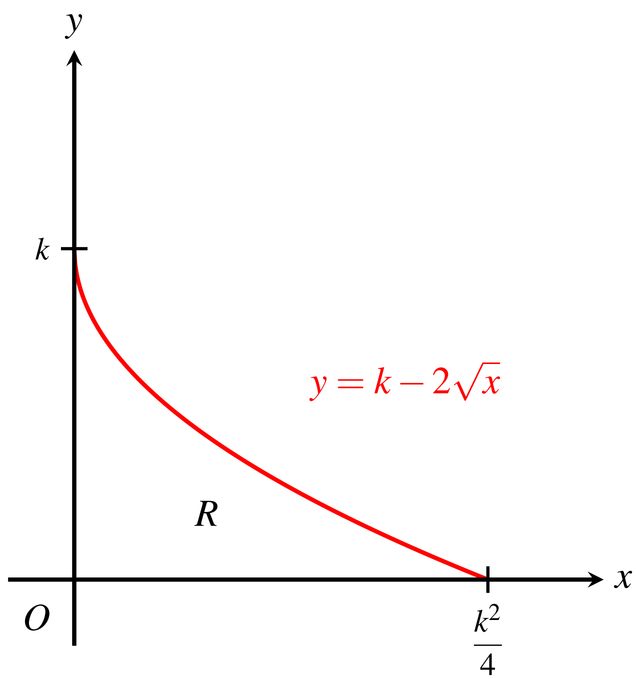
(a) Write, but do not evaluate, an integral that equals the curve's arc length. *(3 pts.)*

(b) Write, but do not evaluate, an integral that equals the surface area generated by rotating the curve about the x-axis. *(3 pts.)*

(c) Write, but do not evaluate, an integral that equals the surface area generated by rotating the curve about the y-axis. *(4 pts.)*

(d) A lamina is the shape bounded by the curve, the y-axis, and the lines $y = 2$ and $x = 2\pi$. Set up integrals for the lamina's centroid $(\bar{x}, \bar{y})$. *(5 pts.)*

16. For $k > 0$, region R is bounded by the graph of $y = k - 2\sqrt{x}$ in the first quadrant, as shown in the following figure.



(a) For $k = 4$, region R is a lamina of density $\frac{1}{2}$. Calculate the lamina's moment about the y-axis. (3 pts.)

(b) For $k = 8$, write an expression involving an integral that equals the perimeter of region R. *(3 pts.)*

(c) The function $p = k - 2\sqrt{x}$ represents a demand curve for $k = 14$. Calculate the consumer surplus if the market price is \$6. *(3 pts.)*

(d) For what value of k is the following function a probability density function? *(4 pts.)*

$$f(x) = \begin{cases} k - 2\sqrt{x} & 0 \leqslant x \leqslant \dfrac{k^2}{4} \\ 0 & \text{otherwise} \end{cases}$$

(e) When $k = 2$, region R has an area of $\frac{2}{3}$ and a centroid at $\left(\frac{3}{10}, \frac{1}{2}\right)$. Let V_x be the volume of the solid generated upon rotating R about the x-axis; similarly, V_y is the volume of the solid obtained by rotating R about the y-axis. Calculate $3V_x + 5V_y$. *(2 pts.)*

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions (5 points each)

1. The quantity produced is the intersection point of the supply curve and the market price:

$$4\sqrt[3]{X+8} = 12 \quad *$$

$$\implies X = 19\,. \quad *$$

Thus, the producer surplus is

$$s_P = \int_0^{19} \left(12 - 4\sqrt[3]{x+8}\right) \mathrm{d}x \quad *$$

$$= \left[12x - 3(x+8)^{4/3}\right]\Big|_0^{19} \quad *$$

$$= 228 - 3(27)^{4/3} + 3(8)^{4/3}$$

$$= \boxed{\$33} \quad *$$

2. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{\sqrt{25-x^2}}\,, \quad *$$

so

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \frac{x^2}{25-x^2} = \frac{25}{25-x^2}\,. \quad *$$

It is most convenient to integrate with x; then the surface area of revolution is

$$S = \int_{-3}^{3} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad *$$

$$= 2\pi \int_{-3}^{3} \sqrt{25 - x^2} \sqrt{\frac{25}{25 - x^2}}\, dx$$

$$= 2\pi \int_{-3}^{3} 5\, dx$$

$$= 10\pi x \Big|_{-3}^{3} \quad *$$

$$= \boxed{60\pi} \quad *$$

3. Differentiating gives

$$\frac{dx}{dy} = \tfrac{1}{2}(y+1)^{1/2}. \quad *$$

So

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \tfrac{1}{4}(y+1) = \tfrac{1}{4}y + \tfrac{5}{4}. \quad *$$

The arc length is

$$L = \int_{-1}^{3} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \quad *$$

$$= \int_{-1}^{3} \sqrt{\tfrac{1}{4}y + \tfrac{5}{4}}\, dy.$$

We substitute

$$u = \tfrac{1}{4}y + \tfrac{5}{4} \implies du = \tfrac{1}{4}\, dy.$$

When $y=-1$, $u=1$; when $y=3$, $u=2$. Hence,

$$\int_{-1}^{3}\sqrt{\tfrac{1}{4}y+\tfrac{5}{4}}\,\mathrm{d}y=\int_{1}^{2}\sqrt{u}\,(4)\,\mathrm{d}u \quad *$$

$$=\tfrac{8}{3}u^{3/2}\Big|_{1}^{2}$$

$$=\boxed{\frac{16\sqrt{2}-8}{3}} \quad *$$

4. The probability is

$$P(x\geqslant 1)=\int_{1}^{\infty}f(x)\,\mathrm{d}x=\int_{1}^{\infty}3x^2e^{-x^3}\,\mathrm{d}x. \quad *$$

We substitute

$$u=-x^3 \implies \mathrm{d}u=-3x^2\,\mathrm{d}x. \quad *$$

When $x=1$, $u=-1$; when $x=\infty$, $u=-\infty$. Thus,

$$P(x\geqslant 1)=\int_{-1}^{-\infty}(-e^u)\,\mathrm{d}u \quad *$$

$$=\int_{-\infty}^{-1}e^u\,\mathrm{d}u.$$

The integral is improper, so we write

$$P(x\geqslant 1)=\lim_{t\to-\infty}\int_{t}^{-1}e^u\,\mathrm{d}u$$

$$=\lim_{t\to-\infty}e^u\Big|_{t}^{-1} \quad *$$

$$=e^{-1}-\lim_{t\to-\infty}e^t$$

$$=\boxed{\frac{1}{e}} \quad *$$

5. To integrate with y, we express all quantities in terms of y. First note that $y = \frac{1}{2}x^2$ can be rewritten as $x = \sqrt{2y}$ (since $0 \leqslant x \leqslant 2$). Then

$$\frac{dx}{dy} = \frac{1}{\sqrt{2y}},$$

so

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{1}{2y} = \frac{2y+1}{2y}. \quad *$$

The surface area of revolution is

$$S = \int_0^1 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy \quad *$$

$$= 2\pi \int_0^1 \sqrt{2y}\sqrt{\frac{2y+1}{2y}}\,dy.$$

$$= 2\pi \int_0^1 \sqrt{2y+1}\,dy.$$

We substitute

$$u = 2y+1 \implies du = 2\,dy.$$

Then $dy = \frac{1}{2}\,du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 3$. Hence,

$$S = \pi \int_1^3 \sqrt{u}\,du \quad *$$

$$= \frac{2\pi}{3}u^{3/2}\Big|_1^3 \quad *$$

$$= \boxed{2\pi\sqrt{3} - \frac{2\pi}{3}} \quad *$$

6. The mean of the probability density function g is

$$\mu = \int_{-\infty}^{\infty} x g(x)\,dx$$

$$= \int_0^{\ln 3} \tfrac{1}{2}xe^x\,dx. \quad *$$

Using Integration by Parts shows

$$\int xe^x \, dx = xe^x - \int e^x \, dx$$

$$= xe^x - e^x + C. \quad *$$

Thus,

$$\mu = \left(\tfrac{1}{2}xe^x - \tfrac{1}{2}e^x\right)\Big|_0^{\ln 3} \quad *$$

$$= \boxed{\tfrac{3}{2}\ln 3 - 1} \quad **$$

7. It is most convenient to let the y-axis face downward such that $y = 0$ is the surface of the water. Then a thin, horizontal strip of the rectangle is located a distance of y beneath the surface, and its width is uniformly 4. Thus,

$$h(y) = y \qquad \text{and} \qquad w(y) = 4. \quad **$$

In addition, the rectangle is bounded between $y = 3$ and $y = 5$. The hydrostatic force is therefore, with $\gamma = 9800$,

$$F = \gamma \int_3^5 h(y)w(y) \, dy$$

$$= 9800 \int_3^5 4y \, dy \quad *$$

$$= 19600y^2 \Big|_3^5 \quad *$$

$$= \boxed{313{,}600 \text{ N}} \quad *$$

8. An exponential probability density function is given by

$$f(t) = \begin{cases} ke^{-kt} & t \geqslant 0 \\ 0 & \text{otherwise}. \end{cases}$$

Let t be the waiting time in minutes. Because 60% of calls are answered within 3 minutes, we have

$$P(0 \leqslant T \leqslant 3) = \int_0^3 ke^{-kt}\,\mathrm{d}t = 0.6 \quad *$$

$$-e^{-kt}\Big|_0^3 = 0.6$$

$$1 - e^{-3k} = 0.6$$

$$\implies k = -\tfrac{1}{3}\ln 0.4\,. \quad *$$

Then the probability that a customer is helped within 5 minutes is

$$P(0 \leqslant T \leqslant 5) = \int_0^5 ke^{-kt}\,\mathrm{d}t \quad *$$

$$= -e^{-kt}\Big|_0^5$$

$$= -(0.4)^{t/3}\Big|_0^5 \quad *$$

$$= \boxed{1 - (0.4)^{5/3}} \quad *$$

9. We take the y-axis to point downward with $y = 0$ being the surface of the water. A thin horizontal strip of the triangle is located a distance

$$h(y) = y \quad *$$

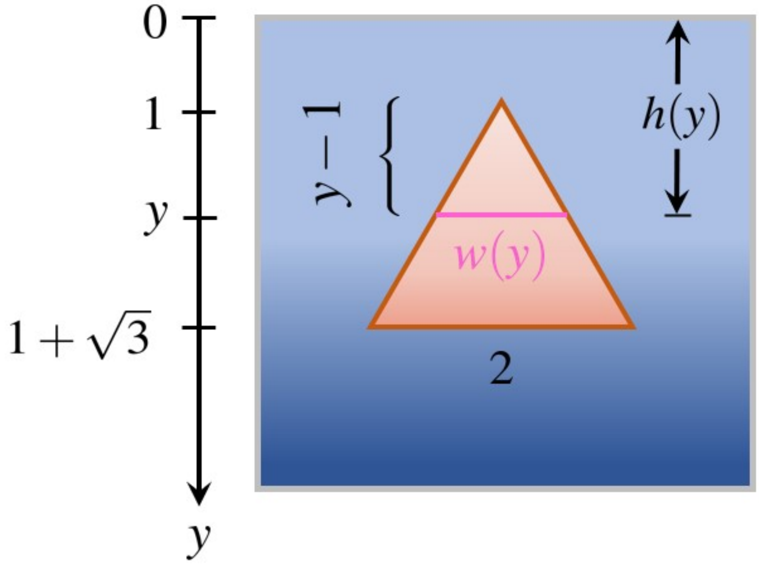
beneath the water's surface. Note that the triangle's height is $\sqrt{3}$. By similar triangles,

$$\frac{y-1}{\sqrt{3}} = \frac{w(y)}{2} \implies w(y) = \frac{2}{\sqrt{3}}(y-1)\,. \quad *$$

The triangle is bounded vertically between $y = 1$ and $y = 1 + \sqrt{3}$. Thus, the hydrostatic force is

$$F = 9800 \int_1^{1+\sqrt{3}} h(y)w(y)\,\mathrm{d}y \qquad *$$

$$= \boxed{9800 \int_1^{1+\sqrt{3}} \frac{2}{\sqrt{3}} y(y-1)\,\mathrm{d}y} \qquad **$$



10. By symmetry, the x-coordinate of the centroid is

$$\bar{x} = 0\,. \qquad *$$

The area of the region (noting that the region is symmetric) is

$$A = \int_{-\pi/3}^{\pi/3} (4\cos x - 1)\,\mathrm{d}x$$

$$= 2\int_0^{\pi/3} (4\cos x - 1)\,\mathrm{d}x$$

$$= 2(4\sin x - x)\Big|_0^{\pi/3}$$

$$= 4\sqrt{3} - \frac{2\pi}{3}\,. \qquad *$$

The y-coordinate of the centroid is

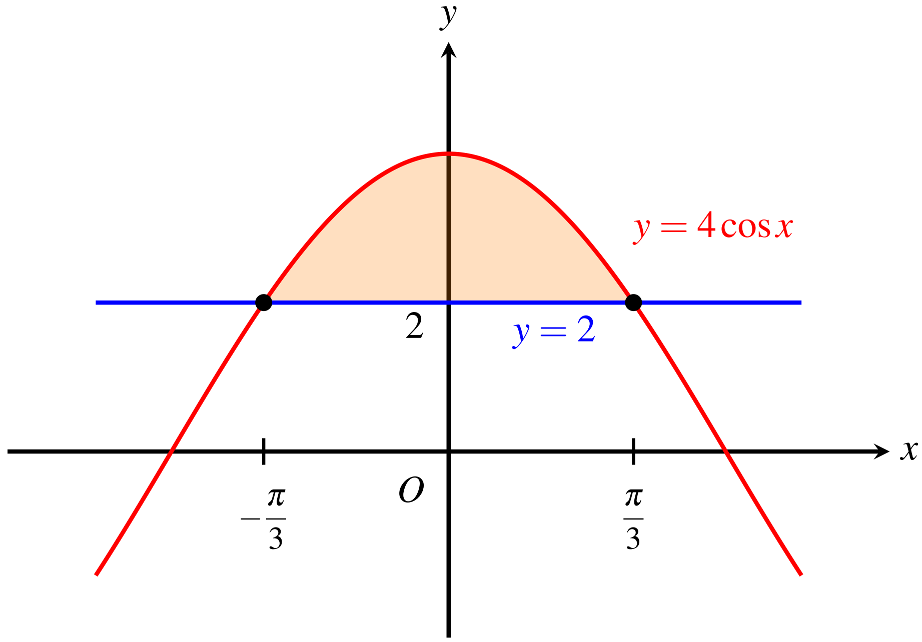
$$\begin{aligned}\bar{y} &= \frac{1}{A}\int_{-\pi/3}^{\pi/3} \tfrac{1}{2}\left[(4\cos x)^2 - (1)^2\right] \mathrm{d}x \\ &= \frac{1}{A}\int_{-\pi/3}^{\pi/3} \tfrac{1}{2}(16\cos^2 x - 1)\,\mathrm{d}x \\ &= \frac{1}{A}\int_{0}^{\pi/3} (16\cos^2 x - 1)\,\mathrm{d}x,\end{aligned}$$ *

where the last step is true by symmetry. Using the reduction formula $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ gives

$$\begin{aligned}\bar{y} &= \frac{1}{A}\int_{0}^{\pi/3} (8 + 8\cos 2x - 1)\,\mathrm{d}x \\ &= \frac{1}{A}\int_{0}^{\pi/3} (7 + 8\cos 2x)\,\mathrm{d}x \\ &= \frac{1}{A}(7x + 4\sin 2x)\Big|_{0}^{\pi/3} \\ &= \frac{\frac{7\pi}{3} + 2\sqrt{3}}{A} \\ &= \frac{7\pi + 6\sqrt{3}}{12\sqrt{3} - 2\pi}.\end{aligned}$$ * *

Hence, the centroid is located at

$$\boxed{\left(0, \frac{7\pi + 6\sqrt{3}}{12\sqrt{3} - 2\pi}\right)}$$



11. Differentiating gives $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 6t^{1/2}$. So

$$1+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 1+36t\,.$$ *

The arc length function is then

$$s(x) = \int_0^x \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$ *

$$= \int_0^x \sqrt{1+36t}\,\mathrm{d}t\,.$$

To evaluate the integral, we substitute

$$u = 1+36t \implies \mathrm{d}u = 36\,\mathrm{d}t\,.$$

When $t = 0$; $u = 1$; when $t = x$, $u = 1 + 36x$. Hence,

$$s(x) = \tfrac{1}{36}\int_1^{1+36x} \sqrt{u}\,\mathrm{d}u \quad *$$

$$= \tfrac{1}{54}u^{3/2}\Big|_1^{1+36x} \quad *$$

$$= \boxed{\frac{(1+36x)^{3/2}-1}{54}} \quad *$$

12. By symmetry, the y-coordinate of the centroid is

$$\bar{y} = \boxed{0} \quad *$$

The curve $x = 10 - y^2$ is formed by the functions

$$y = \sqrt{10-x} \qquad \text{and} \qquad y = -\sqrt{10-x}.$$

As a result, the area is

$$A = \int_1^{10} \left[\sqrt{10-x} - \left(-\sqrt{10-x}\right)\right] \mathrm{d}x$$

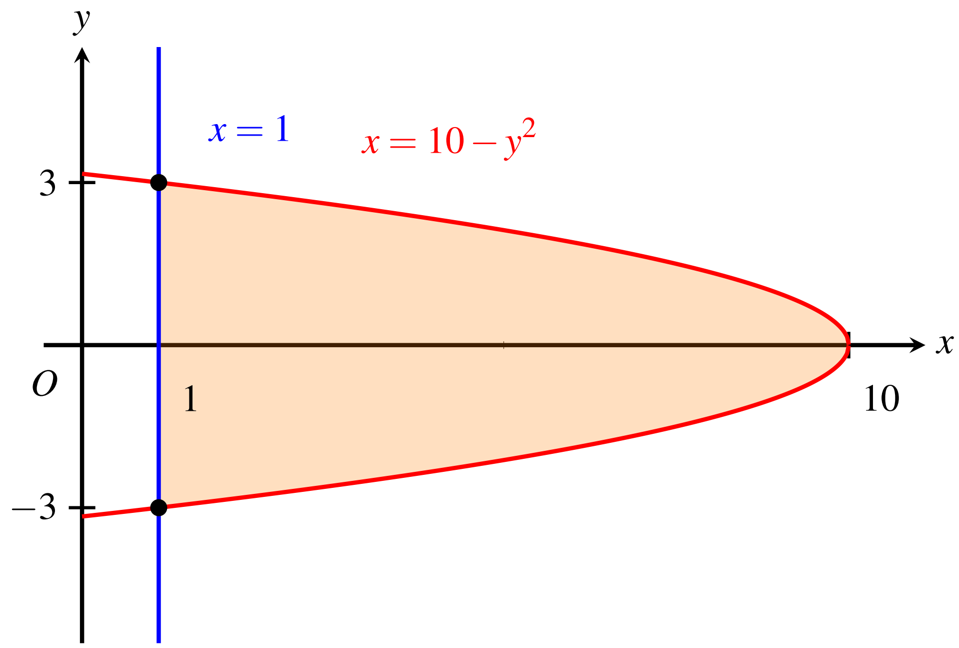
$$= \int_1^{10} 2\sqrt{10-x}\,\mathrm{d}x. \quad *$$

Then the x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_1^{10} x\left[\sqrt{10-x} - \left(-\sqrt{10-x}\right)\right] \mathrm{d}x \quad *$$

$$= \frac{1}{A}\int_1^{10} 2x\sqrt{10-x}\,\mathrm{d}x$$

$$= \boxed{\frac{\int_1^{10} 2x\sqrt{10-x}\,\mathrm{d}x}{\int_1^{10} 2\sqrt{10-x}\,\mathrm{d}x}} \quad **$$



13. The area of the region is

$$A = \int_{-3}^{2} \left[(7 - x^2) - (x + 1)\right] \, dx$$

$$= \int_{-3}^{2} \left(-x^2 - x + 6\right) \, dx .$$

*

The x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A} \int_{-3}^{2} x \left[(7 - x^2) - (x + 1)\right] \, dx$$

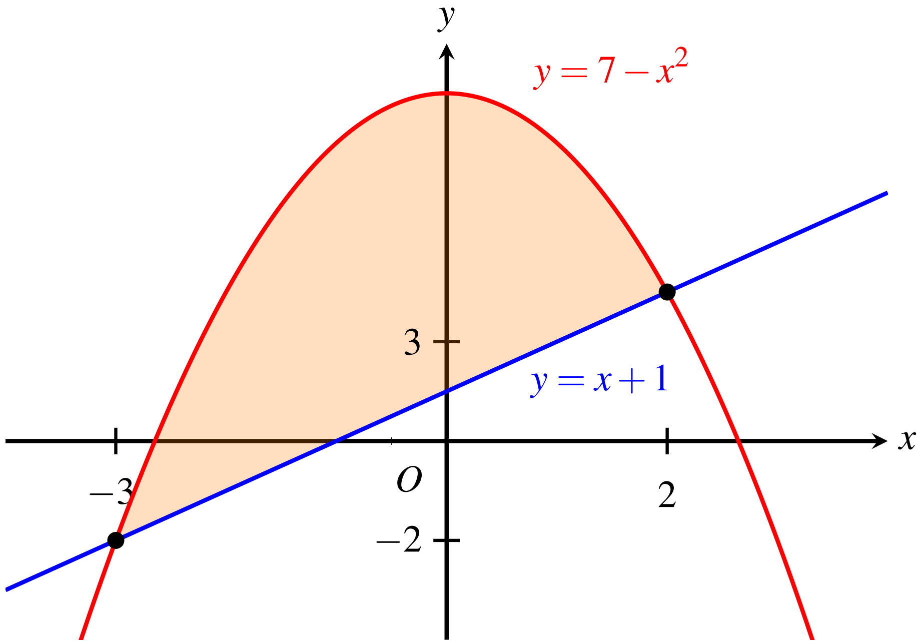
$$= \boxed{\frac{\displaystyle\int_{-3}^{2} x(-x^2 - x + 6) \, dx}{\displaystyle\int_{-3}^{2} \left(-x^2 - x + 6\right) \, dx}}$$

**

Conversely, the y-coordinate is

$$\bar{y} = \frac{1}{A}\int_{-3}^{2} \left[\tfrac{1}{2}(7-x^2)^2 - \tfrac{1}{2}(x+1)^2\right] \mathrm{d}x$$

$$= \boxed{\frac{\int_{-3}^{2} \left[\tfrac{1}{2}(7-x^2)^2 - \tfrac{1}{2}(x+1)^2\right] \mathrm{d}x}{\int_{-3}^{2} \left(-x^2-x+6\right) \mathrm{d}x}}$$ **



14. It is most convenient to select an axis system such that the origin is at the center of the circle. Consider a thin, horizontal strip of the circle. Its depth beneath the water's surface is

$$h(y) = 6 - y,$$ *

and its width is

$$w(y) = 2\sqrt{4-y^2}.$$ *

The disk is bounded from $y = -2$ to $y = 2$ because its radius is 2. Accordingly, the hydrostatic force, using $\gamma = 62.5$, is

$$F = 62.5 \int_{-2}^{2} h(y)w(y)\,\mathrm{d}y \qquad *$$

$$= \boxed{62.5 \int_{-2}^{2} (6-y) \cdot 2\sqrt{4-y^2}\,\mathrm{d}y} \qquad **$$

Long Questions (15 points each)

15. (a) With $\frac{\mathrm{d}y}{\mathrm{d}x} = 3\sin\left(\frac{x}{2}\right)$, we have

$$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+9\sin^2\left(\frac{x}{2}\right). \quad *$$

Thus, the arc length is

$$L = \int_0^{2\pi} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad *$$
$$= \boxed{\int_0^{2\pi} \sqrt{1+9\sin^2\left(\frac{x}{2}\right)}\,\mathrm{d}x} \quad *$$

(b) For revolving around the x-axis, the surface area is

$$S = \int_0^{2\pi} 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad *$$
$$= \boxed{2\pi\int_0^{2\pi} \left[-6\cos\left(\frac{x}{2}\right)+8\right]\sqrt{1+9\sin^2\left(\frac{x}{2}\right)}\,\mathrm{d}x} \quad **$$

(c) For revolving around the y-axis, the simplest integral for the surface area is

$$S = \int_0^{2\pi} 2\pi x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad *$$
$$= \boxed{2\pi\int_0^{2\pi} x\sqrt{1+9\sin^2\left(\frac{x}{2}\right)}\,\mathrm{d}x} \quad ***$$

(d) The area of the lamina is

$$A = \int_0^{2\pi} \left(\left[-6\cos\left(\frac{x}{2}\right)+8\right]-2\right)\mathrm{d}x = \int_0^{2\pi} \left[-6\cos\left(\frac{x}{2}\right)+6\right]\mathrm{d}x. \quad *$$

The x-coordinate of the centroid is

$$\bar{x} = \frac{1}{A}\int_0^{2\pi} x\left(\left[-6\cos\left(\frac{x}{2}\right)+8\right]-2\right)\,\mathrm{d}x \quad *$$

$$= \boxed{\frac{\displaystyle\int_0^{2\pi} x\left[-6\cos\left(\frac{x}{2}\right)+6\right]\,\mathrm{d}x}{\displaystyle\int_0^{2\pi}\left[-6\cos\left(\frac{x}{2}\right)+6\right]\,\mathrm{d}x}} \quad *$$

The y-coordinate of the centroid is

$$\bar{y} = \frac{1}{A}\int_0^{2\pi}\left(\left[-6\cos\left(\frac{x}{2}\right)+8\right]^2-2^2\right)\,\mathrm{d}x \quad *$$

$$= \boxed{\frac{\displaystyle\int_0^{2\pi}\left(\left[-6\cos\left(\frac{x}{2}\right)+8\right]^2-4\right)\,\mathrm{d}x}{\displaystyle\int_0^{2\pi}\left[-6\cos\left(\frac{x}{2}\right)+6\right]\,\mathrm{d}x}} \quad *$$

16. (a) For $y = 4-2\sqrt{x}$, the moment about the y-axis is

$$M_y = \rho\int_0^{k^2/2} x\left(k-2\sqrt{x}\right)\,\mathrm{d}x$$

$$= \tfrac{1}{2}\int_0^4 x\left(4-2\sqrt{x}\right)\,\mathrm{d}x \quad *$$

$$= \int_0^4\left(2x-x^{3/2}\right)\,\mathrm{d}x$$

$$= \left(x^2-\tfrac{2}{5}x^{5/2}\right)\Big|_0^4 \quad *$$

$$= \boxed{\tfrac{16}{5}} \quad *$$

(b) For $y = 8-2\sqrt{x}$, the derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{x}}. \quad *$$

Then the curve's arc length is

$$L = \int_0^{16} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$= \int_0^{16} \sqrt{1 + \left(-\frac{1}{\sqrt{x}}\right)^2}\,\mathrm{d}x$$

$$= \int_0^{16} \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x\,.$$ *

The left side of region R has length k, and the bottom side has length $\frac{k^2}{4}$. Thus, the perimeter of region R is

$$P = k + \frac{k^2}{4} + L$$

$$= 8 + 16 + \int_0^{16} \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x$$

$$= \boxed{24 + \int_0^{16} \sqrt{1 + \frac{1}{x}}\,\mathrm{d}x}$$ *

(c) For $p = 14 - 2\sqrt{x}$, the quantity produced is the intersection of $p = 14 - 2\sqrt{x}$ and the market price, $p = 6$:

$$14 - 2\sqrt{X} = 6$$

$$\implies X = 16\,.$$ *

Thus, the consumer surplus is

$$\begin{aligned} s_C &= \int_0^{16} \left[\left(14 - 2\sqrt{x}\right) - 6\right] \mathrm{d}x \\ &= \int_0^{16} \left(8 - 2\sqrt{x}\right) \mathrm{d}x \\ &= \left(8x - \tfrac{4}{3}x^{3/2}\right)\Big|_0^{16} \\ &= \boxed{\tfrac{128}{3}} \end{aligned}$$ *

(d) The function f is a probability density function if and only if

$$f(x) \geqslant 0 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1\,.$$ *

We see

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \int_0^{k^2/4} \left(k - 2\sqrt{x}\right) \mathrm{d}x \\ &= \left(kx - \tfrac{4}{3}x^{3/2}\right)\Big|_0^{k^2/4} \\ &= \frac{k^3}{12}\,. \end{aligned}$$ *

Equating this expression to 1 shows

$$\begin{aligned} \frac{k^3}{12} &= 1 \\ \implies k &= \boxed{\sqrt[3]{12}} \end{aligned}$$ *

(e) By the Theorem of Pappus,

$$\begin{aligned} V_x &= 2\pi A\,\bar{y} = 2\pi\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{2\pi}{3}\,, \\ V_y &= 2\pi A\,\bar{x} = 2\pi\left(\frac{2}{3}\right)\left(\frac{3}{10}\right) = \frac{2\pi}{5}\,. \end{aligned}$$ *

Then

$$3V_x + 5V_y = 3\left(\frac{2\pi}{3}\right) + 5\left(\frac{2\pi}{5}\right)$$
$$= \boxed{4\pi}$$

*